Circular Symmetry of Pinwheel Diffraction

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Abstract. It is shown that the diffraction of the Conway-Radin pinwheel tiling is circularly symmetric, by explicitly computing the autocorrelation and its Fourier transform.

1 Introduction

The pinwheel tiling was first conceived as a substitution tiling by John H. Conway. Charles Radin later developed the matching rules that determine the same structure [9]. It is an aperiodic tiling of the plane by $1 : 2 : \sqrt{5}$ right triangles and may be constructed by iterating the following substitution rule:

The substitution consists of a standard type of inflation and subdivision rule, and also a second step: a rotation through the angle $\omega := -\arctan \frac{1}{2}$ that aligns the new central triangle with the original tile. This extra step allows repeated applications of the substitution to accumulate into a full tiling that is a fixed point of the substitution.

What makes the pinwheel tiling interesting is that it exhibits tiles of infinitely many orientations, and hence is composed of infinitely many types of tiles in the sense of translational symmetry. In fact its orientations are uniformly distributed over the unit circle (Theorem 3.1). Thus the well-developed theory of tiles and point sets which are of finite local (translational) complexity breaks down. In

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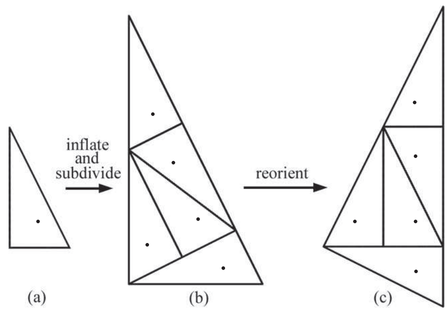

Figure 1: The pinwheel substitution. The figure also indicates the positions of the corresponding control points. The positions of the control points of the full tiling is the set Λ.

particular, the diffraction of the pinwheel tiling (say of its vertices or of control points, one from each tile) is still unknown, even qualitatively.

Progress has been especially hampered by the fact that the number of orientations of the pinwheel grows only linearly in the number of substitutions while the number of tiles is growing exponentially. Thus images derived from computation of the diffraction or autocorrelation turn out to be totally unrepresentative of what is actually happening in the limit.

In this paper, we prove that the diffraction of the pinwheel tiling is circularly symmetric. We do this by explicitly working with the autocorrelation and showing that it, and hence also its Fourier transform (which is by *by definition* the diffraction), converges to a circularly symmetric measure. This process gives some interesting insights as to how the autocorrelation is built out of successive iterations and how the uniform distribution enters into proving its circular symmetry. The result effectively reduces further investigation of the analysis of the pinwheel diffraction to a 1-dimensional problem of the radial autocorrelation. We point out alternative approaches to proving circular symmetry in the last section of the paper.

2 Control Points and the Pinwheel Tiling as a Point Substitution

It is convenient from the point of view of notation and calculation to identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Thus let Γ_0 be the $1 : 2 : \sqrt{5}$ right triangle with vertices $\frac{-1-i}{2}$, $\frac{1-i}{2}$, and $\frac{-1+3i}{2}$. The pinwheel tiling Γ is obtained by iteratively applying the substitution seen in Figure 1 to Γ_0 infinitely many times. Γ_n denotes the set of 5^n tiles obtained by iterating the substitution n times. Tiles that differ from Γ_0 by a Euclidean motion are said to have positive chirality; those that differ by a

Eucidean motion and a reflection possess negative chirality.

Let γ be any tile in Γ. Let u be the vertex at the right angle of γ and let v be the other terminal vertex of the short leg. We define the orientation of γ to be $\theta_\alpha := v - u = e^{i\alpha}$, where α is the angle that the short leg makes to the positive x-axis. Note that θ_α is an element of $U(1) := \{z \mid z \in \mathbb{C}, |z| = 1\}$, the group of rotations around 0 in the plane.

Following [9] we locate a specific distinguished point within a pinwheel tile: if u and v are the vertices defined above and w is the remaining vertex, then the distinguished point of a tile γ is located at $x = \frac{u+2v+w}{4}$. Notice that 0 is the control point of Γ_0 and that 0 occupies the same relative position with respect to the vertices in every supertile Γ_n. It is this property that determines this choice of control points, for it allows us to replace the tiling substitution by a point substitution (see Definition 2.2).

All of the information of a tile γ is encapsulated in its distinguished point, orientation, and chirality, which motivates the following definition:

Definition 2.1 *Let γ be any tile in Γ. The* **control point** *of γ is a triple (x, θ_α, χ) consisting of the distinguished point x of γ, the orientation of γ, and the chirality of γ (± 1) respectively.*

The set of all control points in Γ is denoted by Λ and the set of the control points of Γ_n is Λ_n. By Λ^+, Λ^- we mean the subsets of Λ comprised of the control points of positive and negative chirality, respectively.

Definition 2.2 *The* **pinwheel substitution** *is given by:*

$$(x\,,\theta_\alpha\,,\chi) \mapsto \begin{cases} (\sqrt{5}\theta_\omega x\,,\theta_{\alpha+\omega-\chi\omega}\,,\chi) \\ (\theta_{\alpha+\omega-\chi\omega+\frac{\chi\pi}{2}} + \sqrt{5}\theta_\omega x\,,\theta_{\alpha+\omega-\chi\omega+\pi}\,,\chi) \\ (2\theta_{\alpha+\omega-\chi\omega+\frac{\chi\pi}{2}} + \sqrt{5}\theta_\omega x\,,\theta_{\alpha+\omega-\chi\omega+\pi}\,,-\chi) \\ (\theta_{\alpha+\omega-\chi\omega+\pi} + \sqrt{5}\theta_\omega x\,,\theta_{\alpha+\omega-\chi\omega+\pi}\,,-\chi) \\ (\theta_{\alpha+\omega-\chi\omega-\frac{\chi\pi}{2}} + \sqrt{5}\theta_\omega x\,,\theta_{\alpha+\omega-\chi\omega-\frac{\chi\pi}{2}}\,,-\chi)\,. \end{cases}$$

By infinitely iterating the above substitution on the set starting with the single element $\Lambda_0 = (0\,,\theta_0\,,1)$ we generate Λ. Note that we arbitrarily started with a tile of positive chirality; we could just as easily have used a negative chirality tile. If we repeated the above arguments for the tile $\overline{\Gamma_0}$ (which is the mirror image of Γ_0 in the x-axis), we would obtain a tiling that is a mirror image of the pinwheel tiling. This process would also involve the creation of a mirror substitution. We will use the mirror iterates $V_n := \overline{\Lambda_n}$ in Section 4.2.

In passing we note that it is easy to see that knowing the positions of the control points allows reconstruction of the tiling. It is also the case that knowledge of the positions of the triangles of one chirality determine the positions of the triangles of the other [8].

3 Uniform Distribution of Orientations

Since there is an exact copy of Λ_k in Λ_{k+1} we can define two sequences of angles $\{\alpha_i\}_{i=1}^{\infty}, \{\beta_i\}_{i=1}^{\infty} \subseteq [0, 2\pi)$ such that for any k, $\theta_{\alpha_1}, \dots, \theta_{\alpha_{m_k}}$ are the orientations of the $\chi = 1$ points in Λ_k and $\theta_{\beta_1}, \dots, \theta_{\beta_{n_k}}$ are the orientations of the $\chi = -1$ points. $m_k := \frac{5^k+(-1)^k}{2}$, $n_k := \frac{5^k-(-1)^k}{2}$ are the number of chirality 1, -1 points in Λ_k respectively.

We fix such a sequence for the remainder of the paper.

A key property of the pinwheel tiling is the uniform distribution of the orientations of the tiles [10], in other words the uniform distribution of the two sequences that we have just defined. For the convenience of the reader we provide a short proof of this.

Recall that a sequence $\{z_n\}_{n=1}^{\infty} \subset U(1)$ is **uniformly distributed on** $U(1)$ if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(z_n) = \int_{U(1)} f(z) d\lambda^{U(1)}(z) = \lambda^{U(1)}(f)$$

for all $f : U(1) \to \mathbb{C}$ continuous ($\lambda^{U(1)}$ is normalized Haar measure on $U(1)$). We say that a sequence $\{\gamma_j\}_{j=1}^{\infty} \subset [0, 2\pi)$ is **uniformly distributed modulo** 2π if $\{e^{i\gamma_j}\}_{j=1}^{\infty}$ is uniformly distributed on $U(1)$.

Define

$$M(t) := \begin{pmatrix} e^{it0} + e^{it\pi} & 2e^{it(2\omega-\pi)} + e^{it(2\omega+\frac{\pi}{2})} \\ 2e^{it(\pi)} + e^{it(-\frac{\pi}{2})} & e^{it(2\omega)} + e^{it(2\omega-\pi)} \end{pmatrix} \tag{3.1}$$

for all $t \in \mathbb{Z}$.

If the indices 1 (respectively 2) refer to the tiles of positive (respectively negative) chirality then $(M(1))_{jk}$ is the sum of the orientations of the type j tiles obtained after applying the pinwheel substitution to a single type k tile with orientation 1. Then

$$(M(t))^k = \begin{pmatrix} \sum_{j=1}^{m_k} e^{it\alpha_j} & \sum_{j=1}^{n_k} e^{it(2k\omega-\beta_j)} \\ \sum_{j=1}^{n_k} e^{it\beta_j} & \sum_{j=1}^{m_k} e^{it(2k\omega-\alpha_j)} \end{pmatrix}$$

follows immediately from the definition of $M(t)$ and the pinwheel substitution. [3]

[3] This matrix is similar to the matrix used in [10], [11]. The primary difference comes from the fact that Radin rotates Λ_n at every step so that, considered as one big triangle, it has orientation θ_0. We must use the above matrix in place of Radin's because of our requirement that we work with a fixed point substitution. Also, Radin's type 1 tile corresponds to what we have chosen to be our type 2 tile and vise versa.

Theorem 3.1 (Radin) *$\{\alpha_n\}_n$ and $\{\beta_n\}_n$ are uniformly distributed.*

Proof: By the well-known Weyl criterion a sequence $\{z_n\}_{n=1}^{\infty} \subset U(1)$ is uniformly distributed if and only if for all $t \in \mathbb{Z} \setminus \{0\}$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (z_n)^t = 0\,.$$

Since $\lim\limits_{k\to\infty} \dfrac{m_k}{5^k} = \lim\limits_{k\to\infty} \dfrac{n_k}{5^k} = \dfrac{1}{2}$, it will be enough to prove that, for all $t \neq 0, 1 \leq i, j \leq 2$

$$\lim_{k\to\infty} \frac{((M(t))^k)_{ij}}{5^k} = 0\,.$$

Let $t \neq 0$ be arbitrary but fixed. Let A be the matrix defined by $A_{ij} = |(M(t))_{ij}|$. Then $0 < A \leq \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$ in an entrywise sense, with the additional restriction $A \neq \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$. Also $|((M(t))^k)_{ij}| \leq (A^k)_{ij}$ for all k, i, j. Let λ be the Perron-Frobenius eigenvalue of A. Then, by the Perron-Frobenius Theorem $\lambda < 5$, since 5 is the PF eigenvalue of $\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$. Furthermore there exists a constant c such that for all $n > 0$, $\frac{(A^n)_{ij}}{\lambda^n} \leq c$. Then

$$\left| \frac{((M(t))^k)_{ij}}{5^k} \right| \leq \frac{(A^k)_{ij}}{5^k} = \frac{(A^k)_{ij}}{\lambda^k} \cdot \left(\frac{\lambda}{5}\right)^k \leq c \cdot \left(\frac{\lambda}{5}\right)^k \xrightarrow{k\to 0} 0\,,$$

since $\frac{\lambda}{5} < 1$. Hence, the desired result follows. □

4 Autocorrelation of the Pinwheel Tiling

4.1 Introduction to the Autocorrelation and Some Notation

Let $\mathcal{M}^{\infty}((0,\infty))$ be the subspace of translation bounded [4] regular Borel measures which are supported on $(0,\infty)$ and let $\mathcal{M}(\mathbb{R}^2)$ denote the space of all regular Borel measures on $\mathbb{R}^2$.

Let $\mathcal{P} := \mathbb{C}\backslash\{0\}$, the punctured complex plane and let $\mathcal{M}^*_{pp}(\mathcal{P}) := \big\{ \sum_{k=1}^{n} c_k \delta_{a_k} \big| \; c_k \in \mathbb{R}, n \geq 0, a_k \in \mathcal{P} \big\}$ be the span of all real finitely supported measures on $\mathcal{P}$.

For any $\alpha \in [0, 2\pi)$, we let $R(\alpha) : \mathcal{P} \to \mathcal{P}$ be rotation through angle α: $R(\alpha)(z) := e^{i\alpha} z$. Let σ be the operation of reflection in the x-axis: $\sigma(z) = \overline{z}$.

[4] A measure $\nu \in M(\mathbb{R}^2)$ is **translation bounded** if for all compact $K \subset \mathbb{R}^2$ there exists $c > 0$ such that $|\nu|(a + K) < c$ for all $a \in \mathbb{R}^2$.

Both these types of operations extend to functions and to measures; in particular $R(\alpha)$ and σ act naturally on $\mathcal{M}^*_{pp}(\mathcal{P})$: $R(\alpha)(\sum_{k=1}^{n} c_k\delta_{a_k}) = \sum_{k=1}^{n} c_k\delta_{R(\alpha)(a_k)}$, $\sigma(\sum_{k=1}^{n} c_k\delta_{a_k}) = \sum_{k=1}^{n} c_k\delta_{\overline{a_k}}$

We let $C_c(\mathbb{R}^2)$ denote the space of all continuous compactly supported real-valued functions on $\mathbb{R}^2$.

Let $\lambda^{\mathbb{R}^2}$ be Lebesgue measure on $\mathbb{R}^2$, $\lambda^{U(1)}$ the normalized Haar measure on $U(1)$, and δ_z be the Dirac measure supported at $z \in \mathbb{R}^2$.

A sequence of measures $\{\mu_n\}_n \subset \mathcal{M}(\mathbb{R}^2)$ **converges vaguely** to $\mu \in \mathcal{M}(\mathbb{R}^2)$ if, for all $f \in \mathbb{C}_c(\mathbb{R}^2)$, $\{\mu_n(f)\}_n \overset{n\to\infty}{\longrightarrow} \mu(f)$.

Definition 4.1 *The* **averaged autocorrelation** *of* Λ_n *is*

$$\eta_n := \frac{1}{5^n} \sum_{x,y\in\Lambda_n} \delta_{x-y} \,.$$

The averaged autocorrelation of Λ *is the vague limit* $\eta := \lim_{n\to\infty} \eta_n$ *, if it exists.*[5]

Remark: In defining autocorrelation, one is faced with choosing an averaging sequence, a sequence of compact sets (on which the sums involved are finite) and then taking limits, just as we have done here. For technical reasons, such sequences are chosen to satisfy the van Hove property:

Definition 4.2

(i) For any $A \subset \mathbb{R}^2$ *and* $K \subset \mathbb{R}^2$ *compact, the* **K-boundary** *of* A *is*

$$\partial^K(A) := ((K + A)\backslash A^\circ) \cup ((-K + \overline{\mathbb{R}^2\backslash A}) \cap A),$$

where $^\circ$ *and* $^{-}$ *denote interior and closure, respectively.*

(ii) A **van Hove sequence** *is a sequence of compact subsets* $\{A_n\}_n \subset \mathbb{R}^d$ *such that*

$$\frac{\lambda^{\mathbb{R}^2}(\partial^K(A_n))}{\lambda^{\mathbb{R}^2}(A_n)} \overset{n\to\infty}{\longrightarrow} 0, \quad \textit{for all}\, K \textit{ compact}.$$

In defining the averaged autocorrelation we have used the sequence $\{\Gamma_n\}_{n=1}^{\infty}$ as our averaging van Hove sequence. In Section 5 we will see that we can in fact use any van Hove sequence.

[5]Throughout this section, we will almost exclusively understand Λ to represent only the locations of the control points, which are points of $\mathbb{C} \simeq \mathbb{R}^2$.

4.2 Substitution Formulation for Measures

The primary objective of this section is to verify that η is circularly symmetric. The pinwheel substitution of Definition 2.2 involves complication-causing reflections that we prefer to avoid. Imagine that each tile of some finite part of the pinwheel tiling carries some measure and we are interested in the total sum ν of these measures. We break this total measure into two pieces ν^+ and ν^-, with ν^+ carrying the total measure of the positive chirality tiles and ν^- carrying the measure of the negative chirality tiles *after they have been reflected in the x-axis.* Thus $\nu = \nu^+ + \sigma\nu^-$, but rather than this sum we work with the matrix

$$\begin{pmatrix} \nu^+ \\ \nu^- \end{pmatrix}.$$

In this scheme all measures lie on tiles of positive chirality and the process of reflection can then be relegated to a single operation at the very end that brings the second measure into the correct position. Figure 2 illustrates the formalism as it appears in the substitution process.

Once we have established our formalism, we will use it to generate the η_n. By letting $n \to \infty$, we achieve the desired result.

Definition 4.3 *Let $\Omega, \Phi : (\mathcal{M}^*_{pp}(\mathcal{P}))^2 \to (\mathcal{M}^*_{pp}(\mathcal{P}))^2$ be linear maps defined by:*

$$\begin{aligned}
\Omega\begin{pmatrix} \mu \\ \nu \end{pmatrix} &:= \begin{pmatrix} R(-\omega) & 0 \\ 0 & R(\omega) \end{pmatrix}\begin{pmatrix} \mu \\ \nu \end{pmatrix} \\
\Phi\begin{pmatrix} \mu \\ \nu \end{pmatrix} &:= \frac{1}{5}\begin{pmatrix} R(0)+R(\pi) & 2R(\pi)+R(-\frac{\pi}{2}) \\ 2R(-\pi)+R(\frac{\pi}{2}) & R(-0)+R(-\pi) \end{pmatrix}\begin{pmatrix} \mu \\ \nu \end{pmatrix} \\
\Theta\begin{pmatrix} \mu \\ \nu \end{pmatrix} &:= \begin{pmatrix} id & \sigma \end{pmatrix}\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \mu + \sigma\nu
\end{aligned}$$

Definition 4.4 *For any $k \geq h \geq 1$, define the linear map $\Psi^k_h : (\mathcal{M}^*_{pp}(\mathcal{P}))^2 \to (\mathcal{M}^*_{pp}(\mathcal{P}))^2$ by:*

$$\Psi^k_h\begin{pmatrix} \mu \\ \nu \end{pmatrix} := \Omega^{-k}\Phi\Omega^k\Omega^{-(k-1)}\Phi\Omega^{k-1}\ldots\Omega^{-h}\Phi\Omega^h\begin{pmatrix} \mu \\ \nu \end{pmatrix}.$$

If we define Ψ^m_h to be the identity map whenever $m < h$, then we have $\Psi^k_h\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \Psi^k_l\Psi^{l-1}_h\begin{pmatrix} \mu \\ \nu \end{pmatrix}$ for $k \geq l \geq h \geq 1$. This decomposition of Ψ^k_h will feature in several induction arguments.

Proposition 4.5 *For any $k \geq h \geq 1$,*

$$\Psi_h^k \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \frac{1}{5^{k-(h-1)}} \begin{pmatrix} \sum_{j=1}^{m_{k-(h-1)}} R(\alpha_j) & \sum_{j=1}^{n_{k-(h-1)}} R(2h\omega + \beta_j) \\ \sum_{j=1}^{n_{k-(h-1)}} R(-2h\omega - \beta_j) & \sum_{j=1}^{m_{k-(h-1)}} R(-\alpha_j) \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix}.$$

Proof (by induction on k): Fix an arbitrary $h \geq 1$ for the remainder of this proof.
$k = h$:
$\Psi_h^h \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \Omega^{-h}\Phi\Omega^h \begin{pmatrix} \mu \\ \nu \end{pmatrix}$

$$= \frac{1}{5} \begin{pmatrix} R(0) + R(\pi) & 2R(2h\omega + \pi) + R(2h\omega - \frac{\pi}{2}) \\ 2R(-2h\omega - \pi) + R(-2h\omega + \frac{\pi}{2}) & R(-0) + R(-\pi) \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix}.$$

Induction step: $\Psi_h^{k+1} \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \Psi_{k+1}^{k+1}\Psi_h^k \begin{pmatrix} \mu \\ \nu \end{pmatrix}$. We know what Ψ_{k+1}^{k+1} looks like from our base case above, and we have Ψ_h^k by the induction hypothesis. Because of the symmetry of the Ψ matrices, it is sufficient to consider $(\Psi_h^{k+1})_{11}$ and $(\Psi_h^{k+1})_{12}$:

$$\begin{aligned} (\Psi_h^{k+1})_{11} &= (R(0) + R(\pi)) \sum_{j=1}^{m_{k-(h-1)}} R(\alpha_j) \\ &\quad + (2R(2(k+1)\omega + \pi) + R(2(k+1)\omega - \frac{\pi}{2}) \sum_{j=1}^{n_{k-(h-1)}} R(-2h\omega - \beta_j) \\ &= \sum_{j=1}^{m_{k-(h-1)}} R(\alpha_j) + \sum_{j=1}^{m_{k-(h-1)}} R(\alpha_j + \pi) + \sum_{j=1}^{n_{k-(h-1)}} 2R(2(k-(h-1))\omega - \beta_j + \pi) \\ &\quad + \sum_{j=1}^{n_{k-(h-1)}} R(2(k-(h-1))\omega - \beta_j - \frac{\pi}{2}) = \sum_{j=1}^{m_{(k+1)-(h-1)}} R(\alpha_j). \end{aligned}$$

For help visualizing this argument, see Figure 2. The argument for $(\Psi_h^{k+1})_{12}$ is similar. □

Now that we understand Ψ_h^k in terms of our sequences of angles, we can put the uniform distribution result to good use.

Proposition 4.6 *For any $u \in \mathcal{P}$ and uniformly distributed sequence $\{z_n\}_{n=1}^{\infty} \subset U(1)$ we have:*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} R(z_n)\delta_u = \lambda^{U(1)} \otimes \delta_{|u|},$$

where the limit is in the vague topology.

Proof: Note that the product of measures above refers to the product decomposition $\mathcal{P} = U(1) \times \mathbb{R}_{>0}$.

Let f be any continuous compactly supported $\mathbb{C}$-valued function on $\mathcal{P}$. We are required to show that

$$\lim_{N\to\infty} \left\langle \frac{1}{N}\sum_{n=1}^{N} R(z_n)\delta_u, f \right\rangle = \langle \lambda^{U(1)} \otimes \delta_{|u|}, f\rangle \,.$$

We have

$$\begin{aligned}\left\langle \frac{1}{N}\sum_{n=1}^{N} R(z_n)\delta_u, f \right\rangle &= \frac{1}{N}\sum_{n=1}^{N}\langle \delta_u, R(z_n)^{-1}f\rangle = \frac{1}{N}\sum_{n=1}^{N}\int_{\mathcal{P}} f(z_n x)\, d\delta_u(x) \\ &= \frac{1}{N}\sum_{n=1}^{N} f(z_n u) \overset{N\to\infty}{\longrightarrow} \int_{U(1)} f(zu)dz = \int_{U(1)} f(z\,|u|)dz \\ &= \int_{U(1)\times\mathbb{R}_{>0}} f(zr)\, d\lambda^{U(1)}(z) \otimes \delta_{|u|}(r) = \langle \lambda^{U(1)} \otimes \delta_{|u|}, f\rangle\end{aligned}$$

□

Remark: Note that measures of the form $\lambda^{U(1)} \otimes \sigma$, where σ is a positive measure on $K \subset (0,\infty)$, are not what one may intuitively think from the perspective of usual Lebesgue measure on $\mathbb{R}^2$. For example, consider that $\|\lambda^{U(1)} \otimes \sigma\| = \lambda^{U(1)}(U(1))\sigma(K) = \sigma(K)$. This is independent of where K lies in $(0,\infty)$. The Lebesgue measure of $B_K = U(1) \times K \subset \mathbb{R}^2$ (see Equation (4.3)) is its area, and hence depends on where K is located.

Definition 4.7 *Let $P : \mathcal{P} \to (0,\infty)$ be defined by $P(z) := |z|$. Then P determines a linear map (also denoted by P) from $\mathcal{M}^*_{pp}(\mathcal{P})$ to $\mathcal{M}^\infty((0,\infty))$ by $P(\sum_{k=1}^n c_k\delta_{a_k}) := \sum_{k=1}^n c_k\delta_{P(a_k)}$.*

Corollary 4.8 *For all $\mu,\nu \in \mathcal{M}^*_{pp}(\mathcal{P})$,*

$$\Psi_h^k \begin{pmatrix} \mu \\ \nu \end{pmatrix} \overset{k\to\infty}{\longrightarrow} \frac{1}{2}\begin{pmatrix} \lambda^{U(1)} \otimes (P(\mu+\nu)) \\ \lambda^{U(1)} \otimes (P(\mu+\nu)) \end{pmatrix}$$

in the vague topology.

Proof: The combination of Propositions 4.5 and 4.6 yields

$$\Psi_h^k \begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} \quad \overset{k\to\infty}{\longrightarrow} \quad \frac{1}{2}\begin{pmatrix} \lambda^{U(1)} \otimes \delta_{|x|} + \lambda^{U(1)} \otimes \delta_{|y|} \\ \lambda^{U(1)} \otimes \delta_{|x|} + \lambda^{U(1)} \otimes \delta_{|y|} \end{pmatrix}$$

for any $h \geq 1$. Since μ,ν are finite linear combinations of deltas, the desired result follows by linearity. □

4.3 The Autocorrelation of the nth Iterate

We recall that Λ_n consists of 5 isometrical copies of Λ_{n-1}. Let $n \geq 1$. We define

$$\begin{aligned} D_n &:= \left\{(x,y) \in \Lambda \times \Lambda \,\middle|\, x,y \in \Lambda_n \textit{ and are in different copies of } \Lambda_{n-1}\right\}, \\ C_n &:= \left\{(x,y) \in \Lambda \times \Lambda \,\middle|\, x,y \in \Lambda_n, x \neq y, \textit{ and are in the same copy of } \Lambda_{n-1}\right\}. \end{aligned} \tag{4.1}$$

Then

$$\eta_n = \delta_0 + \frac{1}{5^n} \sum_{x,y \in C_n} \delta_{x-y} + \frac{1}{5^n} \sum_{x,y \in D_n} \delta_{x-y} \,. \tag{4.2}$$

Because the minimum distance between pinwheel control points is $\frac{1}{\sqrt{5}}$, there exists $\frac{1}{\sqrt{5}} > r > 0$ such that $\eta_n|_{B_r(0)} = \delta_0$. For such an r, $\lim\limits_{n\to\infty} \eta_n|_{B_r(0)} = \delta_0$. In other words, 0 is separated from the rest of the support of η.

For any K bounded in $(0,\infty)$, we define

$$B_K := \left\{a \in \mathcal{P} \,\middle|\, |a| \in K\right\} = P^{-1}(K), \tag{4.3}$$

the K-**corona** around 0, whose intersection with the positive x axis is K.

For any $\mu, \nu \in \mathcal{M}^*_{pp}(\mathcal{P})$, we have:

(i) $\mu(B_K) = P(\mu)(K)$,

(ii) $\Psi^k_h \begin{pmatrix} \mu \\ \nu \end{pmatrix} (B_K) = \dfrac{1}{5^{k-(h-1)}} \begin{pmatrix} m_{k-(h-1)}\mu(B_K) + n_{k-(h-1)}\nu(B_K) \\ n_{k-(h-1)}\mu(B_K) + m_{k-(h-1)}\nu(B_K) \end{pmatrix}$.

It is immediate that:

Lemma 4.9 *For all $\nu, \nu' \geq 0$, we have*

$$\Psi^k_h \begin{pmatrix} \nu \\ \nu' \end{pmatrix} (B_K) \leq \begin{pmatrix} \nu(B_K) + \nu'(B_K) \\ \nu(B_K) + \nu'(B_K) \end{pmatrix}.$$

□

Definition 4.10

(i) $\rho_n := \dfrac{1}{5^n} \displaystyle\sum_{x,y \in D_n} \delta_{x-y}$.

(ii) η^+_n, η^-_n *are defined recursively as follows:*
$\eta^+_1 = \eta^-_1 := 0$,
$\begin{pmatrix} \eta^+_n \\ \eta^-_n \end{pmatrix} := \Psi^{n-1}_{n-1} \begin{pmatrix} \eta^+_{n-1} + \rho_{n-1} \\ \eta^-_{n-1} \end{pmatrix}$.

By a standard induction argument, we get

$$\begin{pmatrix} \eta_n^+ \\ \eta_n^- \end{pmatrix} = \sum_{k=1}^{n-1} \Psi_{n-k}^{n-1} \begin{pmatrix} \rho_{n-k} \\ 0 \end{pmatrix} \quad \text{for all}\, n \geq 2\,. \tag{4.4}$$

Proposition 4.11 *For any $n \geq 1$ we have*

$$\eta_n = \delta_0 + \eta_n^+ + \sigma \eta_n^- + \rho_n\,. \tag{4.5}$$

Proof: To see that equation (4.5) holds, by (4.2) we must prove that

$$\eta_{n+1}^+ + \sigma \eta_{n+1}^- = \frac{1}{5^{n+1}} \sum_{(x,y)\in C_{n+1}} \delta_{x-y}\,.$$

We prove this by induction. Figure 2 may help clarify the following argument.
$n=0$: $C_1 = \emptyset$; $\eta_1^+ = 0, \eta_1^- = 0$, which gives us our desired equality.
Induction step: Λ_{n+1} consists of the union of the five disjoint copies of Λ_n resulting from the application of the mappings $f_1, \ldots, f_5$ upon Λ_n. Here f_1, f_2 are direct isometries of $\mathbb{C}$, while f_3, f_4, f_5 are opposite (i.e., chirality reversing) isometries of $\mathbb{C}$ (note that the translation and reflection components of these isometries depend on n, while the rotation components are independent of n). Then,

$$\begin{aligned} C_{n+1} \quad &= \quad \big\{(x,y) \in \Lambda \times \Lambda \,\big|\, \exists\, 1 \leq i \leq 5 \; and \; (a,b) \in \Lambda_n \times \Lambda_n \; with \; a \neq b \\ &\qquad\quad such\ that\ (x,y) = (f_i(a), f_i(b))\big\}\ , \text{so} \end{aligned}$$

The translation part of f_i cancels when we take differences:

$$\begin{aligned} \frac{1}{5^{n+1}} \sum_{(x,y)\in C_{n+1}} \delta_{x-y} \quad &= \quad \frac{1}{5}(R(0) + R(\pi))\Big(\frac{1}{5^n} \sum_{\substack{x,y\in\Lambda_n \\ x\neq y}} \delta_{x-y}\Big) \\ &\quad + \frac{1}{5} R(n\omega)\,\Big(2R(\pi) + R(-\frac{\pi}{2})\Big)\, \sigma R(-n\omega) \Big(\frac{1}{5^n} \sum_{\substack{x,y\in\Lambda_n \\ x\neq y}} \delta_{x-y}\Big)\,. \end{aligned}$$

Now, by the induction hypothesis:

$$\frac{1}{5^n} \sum_{\substack{x,y\in\Lambda_n \\ x\neq y}} \delta_{x-y} = \eta_n - \delta_0 \overset{(3.2)}{=} \eta_n^+ + \rho_n + \sigma \eta_n^-\,.$$

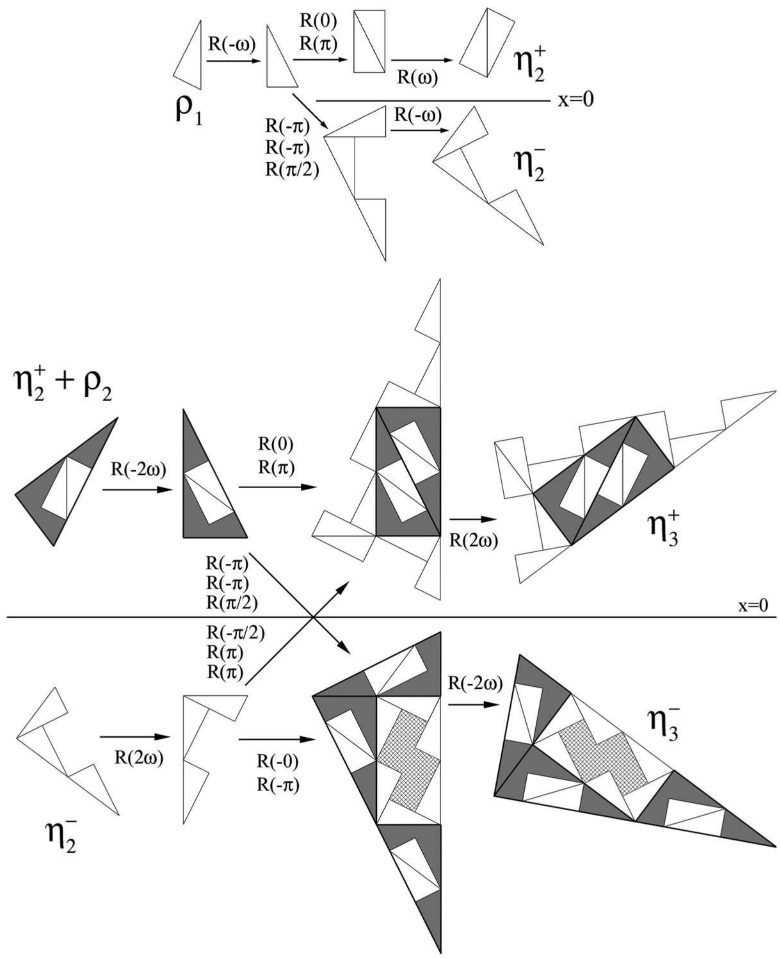

Figure 2: Successive powers of Ψ build up sums of rotation operators which we ultimately apply to measures. Rather than carrying two chiralities in one plane, we prefer to work with one chirality (namely, $+1$) in two planes. These two planes are represented here by the half planes above and below the x-axis respectively. It is to be understood that the full chiral picture is obtained by reflecting the lower half plane onto the upper through the x-axis. The rotations involved are indicated here by the orientation of the triangles which we see being built up by the substitution. After reflection, the upper and lower pictures fit together to give the full substitution.

Therefore,

$$\begin{aligned}
\frac{1}{5^{n+1}} \sum_{(x,y)\in C_{n+1}} \delta_{x-y} &= \frac{1}{5}(R(0)+R(\pi))(\eta_n^+ + \rho_n + \sigma\eta_n^-) \\
&\quad + \frac{1}{5}R(n\omega)(2R(\pi)+R(-\frac{\pi}{2}))\sigma R(-n\omega)(\eta_n^+ + \rho_n + \sigma\eta_n^-) \\
&= \frac{1}{5}\Big((R(0)+R(\pi))(\eta_n^+ + \rho_n) + \sigma(R(-0)+R(-\pi))\eta_n^- \\
&\quad + \sigma(2R(-2n\omega-\pi)+R(-2n\omega+\frac{\pi}{2}))(\eta_n^+ + \rho_n) \\
&\quad + (2R(2n\omega+\pi)+R(2n\omega-\frac{\pi}{2}))\eta_n^-\Big) = \eta_{n+1}^+ + \sigma\eta_{n+1}^-
\end{aligned}$$

by Definition 4.10. □

$$\frac{1}{5^{n+1}} \sum_{(x,y)\in C_{n+1}} \delta_{x-y} = \frac{1}{5}\sum_{i=1}^{5} \frac{1}{5^n} \sum_{\substack{x,y\in\Lambda_n \\ x\neq y}} \delta_{f_i(x)-f_i(y)}\,.$$

4.4 Convergence and Circular Symmetry

Definition 4.12 $\mu_{n,K} := P(\eta_n|_{B_K}) \in \mathcal{M}^\infty((0,\infty))$ *for some* $K\subset(0,\infty)$ *bounded.*

Proposition 4.13 *Let* $K \subseteq (0,\infty)$ *be any bounded set. Then* $\{\mu_{n,K}\}_{n=1}^\infty$ *converges in the total variation norm topology to a pure point measure.*

Proof: By Definition 4.12 and Proposition 4.11, we get

$$\mu_{n+1,K} = P(\eta_{n+1}|_{B_K}) = P(\eta_{n+1}^+|_{B_K}) + P(\eta_{n+1}^-|_{B_K}) + P(\rho_{n+1}|_{B_K}).$$

Note that $P(\eta_{n+1}^-) = P(\sigma\eta_{n+1}^-)$.

By a remark following Equation (4.3), if $K' \subseteq K$ is any set we have

$$\begin{pmatrix} \eta_{n+1}^+(B_{K'}) \\ \eta_{n+1}^-(B_{K'}) \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2\eta_n^+(B_{K'}) + 2\rho_n(B_{K'}) + 3\eta_n^-(B_{K'}) \\ 3\eta_n^+(B_{K'}) + 3\rho_n(B_{K'}) + 2\eta_n^-(B_{K'}) \end{pmatrix},$$

whence

$$\begin{aligned}
&P(\eta_{n+1}^+)(K') + P(\eta_{n+1}^-)(K') = \eta_{n+1}^+(B_{K'}) + \eta_{n+1}^-(B_{K'}) \\
&\quad = \eta_n^+(B_{K'}) + \eta_n^-(B_{K'}) + \rho_n(B_{K'}) = \eta_n(B_{K'}) = P(\eta_n)(K')\,.
\end{aligned}$$

Thus, for all $K' \subseteq K$ we have $P(\eta_{n+1}^+)(K') + P(\eta_{n+1}^-)(K') = P(\eta_n)(K')$. Hence,

$$\begin{aligned}
P(\eta_{n+1}^+|_{B_K}) + P(\eta_{n+1}^-|_{B_K}) &= P(\eta_{n+1}^+)|_K + P(\eta_{n+1}^-)|_K \\
&= P(\eta_n)|_K = P(\eta_n|_{B_K}) = \mu_{n,K}\,. \qquad (4.6)
\end{aligned}$$

So we get
$$\mu_{n+1,K} = \mu_{n,K} + P(\rho_{n+1}|_{B_K}) \,.$$
Therefore, $\mu_{n+1,K} \geq \mu_{n,K}$ and
$$\|\mu_{n+1,K} - \mu_{n,K}\| = \rho_{n+1}(B_K) = \frac{1}{5^{n+1}}\mathrm{card}\big\{(x,y) \in D_{n+1} \bigm| |x-y| \in K\big\}.$$
Because x, y must be in different copies of Λ_n, x must be in the B_K-boundary of one of those copies and $y \in x + B_K$. Let $c := \max\limits_{a\in\mathbb{C}}(\mathrm{card}\{\Lambda \cap (a + B_K)\})$, a finite quantity because the minimum distance between control points is $\frac{2}{\sqrt{5}}$. Then,
$$\mathrm{card}\big\{(x,y) \in D_{n+1} \bigm| |x-y| \in K\big\} \leq c \cdot \sum_{j=1}^{5} \mathrm{card}\big\{x \in \Lambda \cap \partial^{B_K}(f_j\Gamma_n)\big\} \,.$$

When we inflate Γ_n we have $\lambda^{\mathbb{R}^2}(\partial^{B_K}\Gamma_{n+1}) \simeq \sqrt{5}\lambda^{\mathbb{R}^2}(\partial^{B_K}\Gamma_n)$, since the linear scaling is by $\sqrt{5}$.

Therefore, $\exists$ a constant c' depending only on K such that
$$\rho_{n+1}(B_K) \leq c' \left(\frac{1}{\sqrt{5}}\right)^{n+1} \,. \tag{4.7}$$
Then $\|\mu_{m,K} - \mu_{n,K}\| \leq c' \sum_{j=n+1}^{m}(\frac{1}{\sqrt{5}})^j$ shows that $\{\mu_{n,K}\}_n$ is Cauchy in the total variation norm. By a comment following Proposition 3 of [1], $\{\mu_{n,K}\}_n$ converges in the total variation norm topology to a pure point measure. $\square$

Definition 4.14 $\mu_K := \lim\limits_{n\to\infty} \mu_{n,K}$ *is a pure point measure on* $(0,\infty)$.

Proposition 4.15 $\eta_n|_{B_K} \longrightarrow \lambda^{U(1)} \otimes \mu_K$ *in the vague topology.*

Proof: Let $\eta_K = \lambda^{U(1)} \otimes \mu_K$. Let U be any neighbourhood of 0 in the vague topology. Then $\exists\, V$, a neighbourhood of 0, such that $V + V + V + V + V + V \subseteq U$. Also, we may assume that $V = -V$. Since the total variation topology is stronger than the vague topology, there exists $\epsilon > 0$ such that whenever $\|\nu\| < \epsilon$ then $\nu \in V$.

Because $\mu_{n,K} \xrightarrow{\|\cdot\|} \mu_K$, there exists N such that for all $n > N$, we have $\|\mu_{n,K} - \mu_K\| < \epsilon$. This gives us $\|\lambda^{U(1)} \otimes \mu_{n,K} - \lambda^{U(1)} \otimes \mu_K\| < \epsilon$, and hence, $\lambda^{U(1)} \otimes \mu_{n,K} - \eta_K \in V$ for all $n > N$.

(4.7) says that $\rho_n(B_K) \leq c' \left(\frac{1}{\sqrt{5}}\right)^n$, so $\exists\, M \geq N + 1$ such that
$$\sum_{k=M}^{m} \rho_k(B_K) < \epsilon \text{ for all } m \geq M. \tag{4.8}$$

We know by (4.4) that

$$\begin{pmatrix} \eta_n^+|_{B_K} \\ \eta_n^-|_{B_K} \end{pmatrix} = \sum_{k=1}^{n-1} \Psi_{n-k}^{n-1} \begin{pmatrix} \rho_{n-k}|_{B_K} \\ 0 \end{pmatrix} . \tag{4.9}$$

Splitting the above sum yields

$$\begin{pmatrix} \eta_{n+M}^+|_{B_K} \\ \eta_{n+M}^-|_{B_K} \end{pmatrix} - \Psi_M^{n+M-1} \begin{pmatrix} \eta_M^+|_{B_K} \\ \eta_M^-|_{B_K} \end{pmatrix} = \sum_{k=1}^{n} \Psi_{n+M-k}^{n+M-1} \begin{pmatrix} \rho_{n+M-k}|_{B_K} \\ 0 \end{pmatrix} ,$$

and using the triangle inequality gets us

$$\begin{aligned} \left\| \begin{pmatrix} \eta_{n+M}^+|_{B_K} \\ \eta_{n+M}^-|_{B_K} \end{pmatrix} - \Psi_M^{n+M-1} \begin{pmatrix} \eta_M^+|_{B_K} \\ \eta_M^-|_{B_K} \end{pmatrix} \right\| &\le \sum_{k=1}^{n} \left| \Psi_{n+M-k}^{n+M-1} \begin{pmatrix} \rho_{n+M-k} \\ 0 \end{pmatrix} \right| (B_K) \\ &= \sum_{k=1}^{n} \Psi_{n+M-k}^{n+M-1} \begin{pmatrix} \rho_{n+M-k} \\ 0 \end{pmatrix} (B_K) , \end{aligned}$$

where $\left\| \begin{pmatrix} \nu \\ \nu' \end{pmatrix} \right\| := \begin{pmatrix} \|\nu\| \\ \|\nu'\| \end{pmatrix}$ and $\|\cdot\|$ is the total variation norm.

Thus, by Lemma 4.9 and (4.8),

$$\left| \begin{pmatrix} \eta_{n+M}^+ \\ \eta_{n+M}^- \end{pmatrix} - \Psi_M^{n+M-1} \begin{pmatrix} \eta_M^+ \\ \eta_M^- \end{pmatrix} \right| (B_K) < \begin{pmatrix} \epsilon \\ \epsilon \end{pmatrix} . \tag{4.10}$$

We also know that $\lambda^{U(1)} \otimes \mu_{M-1} - \eta_K \in V$. From Corollary 4.8 and the fact that M is fixed, we know

$$\Psi_M^{n+M-1} \begin{pmatrix} \eta_M^+|_{B_K} \\ \eta_M^-|_{B_K} \end{pmatrix} \overset{n\to\infty}{\longrightarrow} \frac{1}{2} \begin{pmatrix} \lambda^{U(1)} \otimes P(\eta_M^+|_{B_K} + \eta_M^-|_{B_K}) \\ \lambda^{U(1)} \otimes P(\eta_M^+|_{B_K} + \eta_M^-|_{B_K}) \end{pmatrix}$$

and so, by (4.6), we get that

$$\Psi_M^{n+M-1} \begin{pmatrix} \eta_M^+|_{B_K} \\ \eta_M^-|_{B_K} \end{pmatrix} \overset{n\to\infty}{\longrightarrow} \frac{1}{2} \begin{pmatrix} \lambda^{U(1)} \otimes \mu_{M-1} \\ \lambda^{U(1)} \otimes \mu_{M-1} \end{pmatrix} .$$

Therefore, $\exists\, N'$ such that

$$\Psi_M^{n+M-1} \begin{pmatrix} \eta_M^+|_{B_K} \\ \eta_M^-|_{B_K} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \lambda^{U(1)} \otimes \mu_{M-1} \\ \lambda^{U(1)} \otimes \mu_{M-1} \end{pmatrix} \in \begin{pmatrix} V \\ V \end{pmatrix} \quad \text{for all}\, n \ge N' ,$$

and by (4.10), we have

$$\begin{pmatrix} \eta_{n+M}^+|_{B_K} \\ \eta_{n+M}^-|_{B_K} \end{pmatrix} - \Psi_M^{n+M-1} \begin{pmatrix} \eta_M^+|_{B_K} \\ \eta_M^-|_{B_K} \end{pmatrix} \in \begin{pmatrix} V \\ V \end{pmatrix} \quad \text{for all}\, n \ge 0 .$$

Combining these, we get

$$\begin{pmatrix} \eta^+_{n+M}|_{B_K} \\ \eta^-_{n+M}|_{B_K} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \lambda^{U(1)} \otimes \mu_{M-1} \\ \lambda^{U(1)} \otimes \mu_{M-1} \end{pmatrix} \in \begin{pmatrix} V - V \\ V - V \end{pmatrix} \quad \text{for all}\, n \geq N' \, .$$

Finally, from Proposition 4.11

$$\begin{aligned} \eta_{n+M}|_{B_K} - \eta_K \quad &= \quad (\eta^+_{n+M}|_{B_K} - \frac{1}{2}\lambda^{U(1)} \otimes \mu_{M-1}) + (\eta^-_{n+M}|_{B_K} - \frac{1}{2}\lambda^{U(1)} \otimes \mu_{M-1}) \\ &\quad + (\lambda^{U(1)} \otimes \mu_{M-1} - \lambda^{U(1)} \otimes \mu_K) + \rho_{n+M}|_{B_K} \end{aligned}$$

and then (4.8) gives us $\|\rho_{n+M}|_{B_K}\| < \epsilon$ whence $\rho_{n+M}|_{B_K} \in V$, by our choice of ϵ. Thus

$$\eta_{n+M}|_{B_K} - \eta_K \in V - V + V - V + V + V \subseteq U \, ,$$

and therefore $\eta_n|_{B_K} - \eta_K \in U$ for all $n > N' + M$. □

4.5 Autocorrelation Conclusions

It is easy to see that if $K \subseteq K'$ then $\mu_{K'}|_K = \mu_K$. This allows the following definition:

Definition 4.16 *μ is the pure point measure on $(0, \infty)$ defined by $\mu|_K = \mu_K$ for all K bounded in $(0, \infty)$.*

From Proposition 4.15 and the fact that $\lim_{n\to\infty} \eta_n|_{B_r(0)} = \delta_0$ for some sufficiently small $r > 0$, we get that $\eta_n|_{\{0\}\cup B_K} \longrightarrow \delta_0 + \lambda^{U(1)} \otimes \mu|_K$ for all $K \subseteq (0, \infty)$ bounded. This final remark sets us up for the main theorem.

Theorem 4.17 *The autocorrelation of Λ, η, exists with respect to $\{\Lambda_n\}_{n=1}^\infty$ and $\eta = \delta_0 + \lambda^{U(1)} \otimes \mu$.*

Proof: Suppose that f is an arbitrary real valued continuous function of compact support. Then, $\mathrm{supp}(f) \subseteq \{0\} \cup B_K$ for some bounded $K \subset (0, \infty)$.

From Proposition 4.15 we have

$$\eta_n|_{\{0\}\cup B_K} \longrightarrow \delta_0 + \lambda^{U(1)} \otimes \mu_K \, ,$$

which means

$$\eta_n|_{\{0\}\cup B_K}(f) \longrightarrow (\delta_0 + \lambda^{U(1)} \otimes \mu_K)(f).$$

Because $\mathrm{supp}(f) \subseteq \{0\} \cup B_K$, this gives us

$$\eta_n(f) \longrightarrow (\delta_0 + \lambda^{U(1)} \otimes \mu)(f) \, ,$$

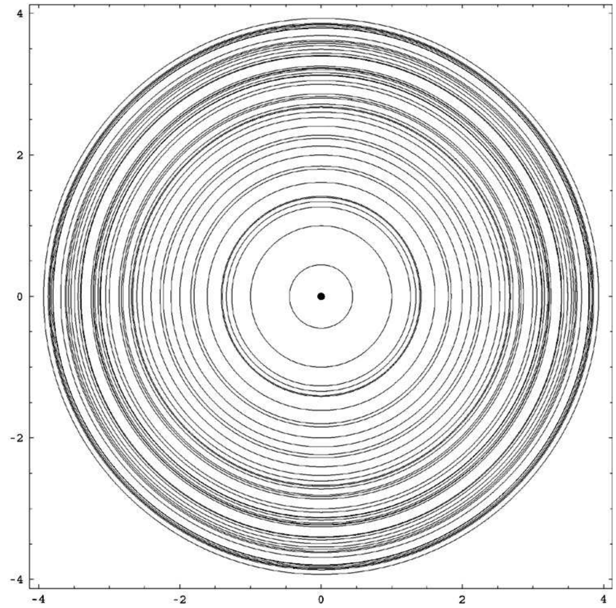

Figure 3: Part of the support of the pinwheel autocorrelation measure η

and finally, by the definition of vague convergence,

$$\eta_n \longrightarrow \delta_0 + \lambda^{U(1)} \otimes \mu\,.$$

$\square$

Our understanding of the autocorrelation of the pinwheel tiling only lacks knowledge regarding the pure point measure μ, and hence about the radii and heights of the circles. In [10], Charles Radin suggests that the support of μ has a self-similar structure. While we were not able to exploit this observation, it may prove useful to future pinwheel enthusiasts.

4.6 Diffraction

The diffraction of an object is the Fourier transform of its autocorrelation measure. One may consult [6], [7] for the diffraction theory and [3] for the theory of translation bounded measures and their Fourier transforms. Note that η_n, η are all translation bounded, positive-definite, and positive measures ([2], Proposition 7).

Recall that for all $f \in C_c(\mathbb{R}^2)$ and $n \geq 1$, $\langle \widehat{\nu}, f \rangle = \langle \nu, \widehat{f} \rangle$.

Since the Fourier transformation is a homeomorphism on the set of all positive and positive-definite measures on $\mathbb{R}^2$ equipped with the vague topology

([3],Theorem 4.16), from $\{\eta_n\}_n \stackrel{n\to\infty}{\longrightarrow} \eta$ in the vague topology, we know that $\{\widehat{\eta_n}\}_n$ also converges vaguely as $n \to \infty$. Thus the **diffraction** of the pinwheel tiling is the Fourier transform of its autocorrelation, $\widehat{\eta} := \lim\limits_{n\to\infty} \widehat{\eta_n}$.

It is easy to check that for $R(\alpha) \in U(1)$, $f \in C_c(\mathbb{R}^2)$ we have: $(R(\alpha)\widehat{f})(k) = \widehat{R(\alpha)f}(k)$ and hence, if ν is a Fourier transformable measure and $R(\alpha)\nu = \nu$ then $\widehat{\nu}(R(\alpha)f) = \widehat{\nu}(f)$. Since the pinwheel autocorrelation is fully circularly symmetric, so is the diffraction.

From this, we can see that the diffraction may only have a pure point part at the origin. We show that this is indeed the case.

Proposition 4.18 $\widehat{\eta}_{pp} = \delta_0$.

Proof: This result follows from Theorem 2.2 in [7]. We have

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(\Gamma_n)} \int 1\, d(\sum_{x\in\Lambda_n} \delta_x) = \lim_{n\to\infty} \frac{\mathrm{card}(\Lambda_n)}{\mathrm{vol}(\Gamma_n)} = \mathrm{dens}(\Lambda)\,.$$

Then, by the result mentioned above, $\widehat{\eta}(\{0\}) = (\mathrm{dens}(\Lambda))^2 = 1$. □

5 van Hove sequences

In the construction of the autocorrelation and the diffraction we have assumed that the averaging sequence is the ascending chain of super triangles Γ_n created by the substitution process itself. In this section we prove that we get the same autocorrelation (and hence diffraction) for any van Hove sequence $\mathcal{A} = \{A_m\}$.

Proposition 5.1 *Let $\mathcal{A} = \{A_m\}$ be any van Hove sequence in $\mathbb{R}^2$ and let*

$$\eta_{A_m} := \frac{1}{\mathrm{vol}\,(A_m)} \sum_{x,y\in\Lambda\cap A_m} \delta_{x-y} \tag{5.1}$$

be the averaged autocorrelation of A_m. Then $\eta_{A_m} \to \eta$ in the vague topology.

Proof: It suffices to take one fixed, but otherwise arbitrary, continuous function f on $\mathbb{R}^2$ of compact support and show that $\eta_{A_m}(f) \to \eta(f)$. We assume $f \neq 0$.

We know that $\eta_n \to \eta = \lambda^{U(1)} \otimes \mu$ in the vague topology. Since $\lambda^{U(1)} \otimes \mu = \sigma(\lambda^{U(1)} \otimes \mu)$, we also have that $\sigma\eta_n \to \eta$.

Let B be a closed ball around 0 which contains $\mathrm{supp}(f)$. Since $R_\theta(\eta) = \eta$ for all θ, it follows that $R_\theta\eta_n(f) \to \eta(f)$ for each θ. Since the mapping $(\theta, x) \mapsto f(R_\theta x)$ is continuous and $U(1) \times B$ is compact we see that the convergence to $\eta(f)$ is uniform in θ. In the same way $R_\theta(\sigma\eta_n) \to \eta$ uniformly for all θ.

Consider any of the triangles Γ_n and in particular its inner boundary of width equal to the diameter of the ball B. Let $p(\Gamma_n)$ be the perimeter of Γ_n. Since Λ is a Delone set, there is a positive constant c_B so that the number of points of Λ inside this inner boundary is bounded above by $c_B p(\Gamma_n)$ for all n, no matter where or in what orientation the triangle Γ_n is placed in $\mathbb{R}^2$. Let n_B be the maximum number of points of Λ inside any translate of B.

Combining all this information, for any $\epsilon > 0$ we can choose $N = N(\epsilon, f) > 0$ so that for all θ and for all $k \geq 0$ we have simultanously

- $|R_\theta \eta_N(f) - \eta(f)| < \epsilon$
- $|R_\theta \sigma \eta_N(f) - \eta(f)| < \epsilon$
- $\dfrac{c_B p(\Gamma_N) n_B}{\mathrm{vol}\,(\Gamma_N)} ||f||_\infty < \epsilon$.

Let A be any region of $\mathbb{R}^2$ precisely tiled by a subset of the super-tiles Γ_N in the total tiling Γ of $\mathbb{R}^2$. Then

$$A = \bigcup_{i=1}^{M} T_i \Gamma_N$$

where the T_i are composed of Euclidean isometries, and since autocorrelations are unaffected by translations,

$$\left| \frac{1}{M} \sum_{i=1}^{M} \eta_{T_i \Gamma_N}(f) - \eta(f) \right| \leq \frac{1}{M} \sum_{i=1}^{M} |\eta_{T_i \Gamma_N}(f) - \eta(f)| \leq \epsilon \ . \tag{5.2}$$

The averaged autocorrelation of η_A of A is

$$\eta_A(f) = \frac{1}{M} \sum_{i=1}^{M} \eta_{T_i \Gamma_N}(f) + \frac{1}{\mathrm{vol}\, A} \sum_{(x,y) \in D(A,N)} f(x-y) \tag{5.3}$$

where $D(A, N)$ is the set of all pairs $(x, y) \in (A \cap \Lambda) \times (A \cap \Lambda)$ where the two components come from different copies of the tile Γ_N in its tiling of A.

Since $x - y \in B$ is necessary for (x, y) to make any contribution to the sum, x is restricted to the inner B-boundary of the tile it belongs to and y is restricted to the ball $x + B$. Thus

$$\left| \frac{1}{\mathrm{vol}\, A} \sum_{(x,y) \in D(A,N)} \delta_{x-y}(f) \right| < \frac{M c_B p(\Gamma_N) n_B ||f||_\infty}{M \mathrm{vol}(\Gamma_N)} < \epsilon \ . \tag{5.4}$$

Thus

$$|(\eta_A - \eta)(f)| < 2\epsilon \ . \tag{5.5}$$

Now consider the van Hove sequence $\{A_m\}$. Let K be any closed disk centred on 0 containing Γ_N and $\overline{\Gamma_N}$. Let $A(m)$ be that part of A_m which is composed of complete Γ_N – tiles taken from the full tiling Γ. Then $A_m \backslash A(m) \subset \partial^K(A_m)$.

Now the point is that because of the van Hove property the boundary can contain only a number of points of Λ that is bounded by $c_K \mathrm{vol}(\partial^K A_m)$ for some positive constant c_K that is independent of m. Thus

$$\begin{aligned}\left|\eta_{A_m}(f) - \frac{\mathrm{vol}\, A(m)}{\mathrm{vol}\, A_m}\eta_{A(m)}(f)\right| &= \left|\frac{1}{\mathrm{vol}\, A_m}\sum \delta_{x,y}(f)\right| \\ &\leq \frac{c_1}{\mathrm{vol}\, A_m}\mathrm{vol}\,(\partial^K A_m)||f||_\infty\end{aligned} \tag{5.6}$$

where the sum is over all $x, y \in \Gamma \cap A_m$ in which at least one of x or y is in $\partial^K(A_m)$. The van Hove property shows that $\eta_{A(m)} - \eta_{A_m} \to 0$ as $m \to \infty$.

Combining this with equation (5.5) we see that $|\eta(f) - \eta_{A_m}(f)| < 3\epsilon$ for all $m >> N = N(\epsilon, f)$. As ϵ was arbitrary and so was f, we have demonstrated the proposition. □

6 Further Remarks

We point out that there are (at least) two other approaches to circular symmetry. For these we consider the space X of all tilings that are locallly indistinguishable from the pinwheel tiling that we have constructed. These are the tilings each of whose patches is a copy, under the rotation-translation group of the full Euclidean group of isometries of the plane, to a patch of the given pinwheel tiling, and vica-versa. This is evidently closed under the rotation-translation group and in particular under the translation group of the plane. We give this space the standard topology [13]. The uniform distribution of orientations allows us to conclude that X is the closure of the *translation* orbit of any of its tilings – it is minimal.

The substitution together with the uniform distribution of orientation allows one to see quite easily that patch frequencies are uniform – the limit defining the frequency, or density, of a patch of tiles is approached uniformly, independent of the position or orientation of the patch. It follows from this that the autocorrelation is identical for all elements of X, and so this measure must be circularly symmetric.

Alternatively one can try to approach the problem by looking at the dynamical spectrum arising from the unitary action of $\mathbb{R}^2$ on $L^2(X, \mathbb{R}^2, \mu)$, where μ is the unique ergodic measure on X. The relationship between this and the diffraction is given by a piece of formalism called Dworkin's argument [4]. The exact scope of this relationship is not understood, but it has been successfully used to infer things about the diffraction spectrum, particularly in the case of pure point diffraction. It was, for instance, used in the first proofs of the pure point diffractiveness of regular model sets. Now, it is the case that in the case of the pinwheel tiling the dynamical spectrum is circularly symmetric [12] , and so it is conceivable that some form of Dworkin's argument might be developed which would allow this result to be carried over to the diffraction. To our knowledge this has not been explicitly carried out.

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